Data Assimilation
- A06. Ensemble Kalman Filter -

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Environmental Prediction Science Laboratory

DA Lectures A (Basic Course)

- (1) Introduction and NWP
- (2) Deterministic Chaos and Lorenz-96 model
- (3) A toy model and Bayesian estimation
- (4) Kalman Filter (KF)
- (5) 3D Variational Method (3DVAR)
- (6) Ensemble Kalman Filter (PO method)
- (7) Serial Ens. Square Root Filter (Serial EnSRF)
- (8) Local Ens. Transform Kalman Filter (LETKF)
- (9) Innovation Statistics
- (10) Adaptive Inflations
- (11) 4D Variational Method (4DVAR)

Today's Goal

- **Lecture: Ensemble Kalman Filter**
  - to introduce EnKF
  - to understand PO method

- **Training Course: Lorenz 96**
  - to implement PO method
  - to implement localization

Ensemble Kalman Filter (EnKF)

Why EnKF?

Kalman Filter

$$\mathbf{P}_t^b \quad (n \times n)$$

Background error covariance cannot be stored on RAMfor high dimensional models such as NWP (n~O(10^{12}~10^{15})

Ex) if $n=10^6$ ➔ 10^{12} x 8 byte = 8 TB

Ensemble Kalman Filter

$$\mathbf{P}_t^b \approx \frac{\delta\mathbf{X}_t^b\left(\delta\mathbf{X}_t^b\right)^T}{m-1}$$

$$\delta\mathbf{X}_t^b \quad (n \times m)$$

An approximation of error covariance with ensemble perturbation matrix $\delta\mathbf{X}$.

m: ensemble size

Conceptual Images

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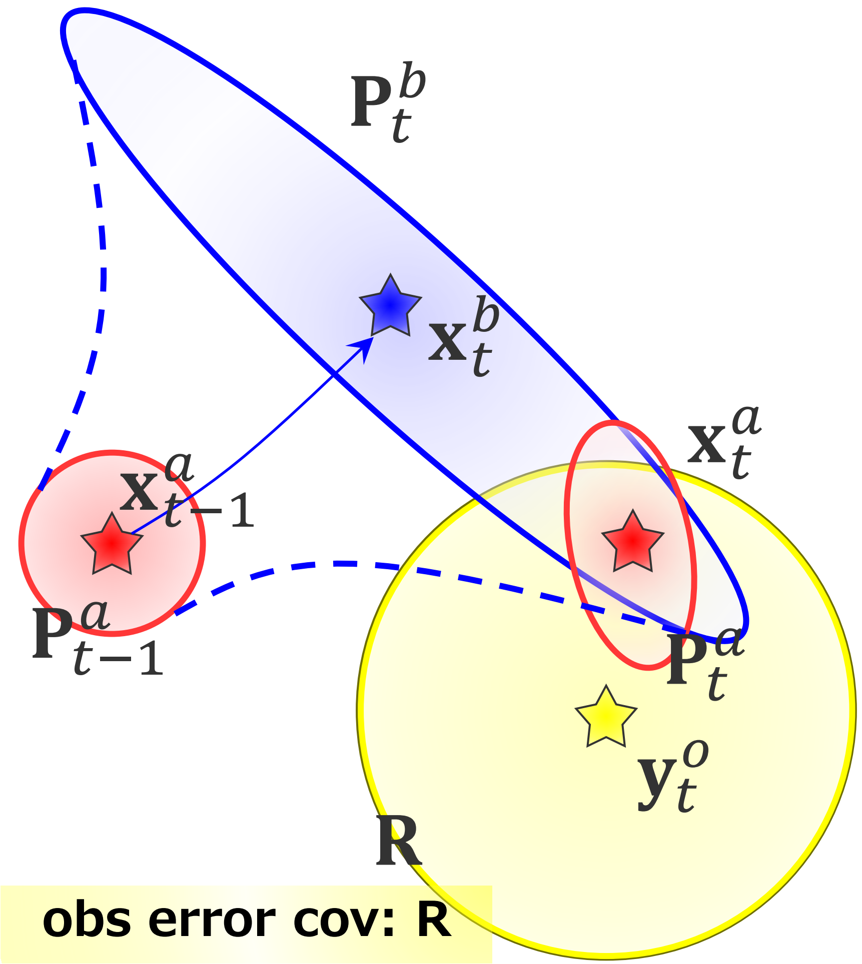
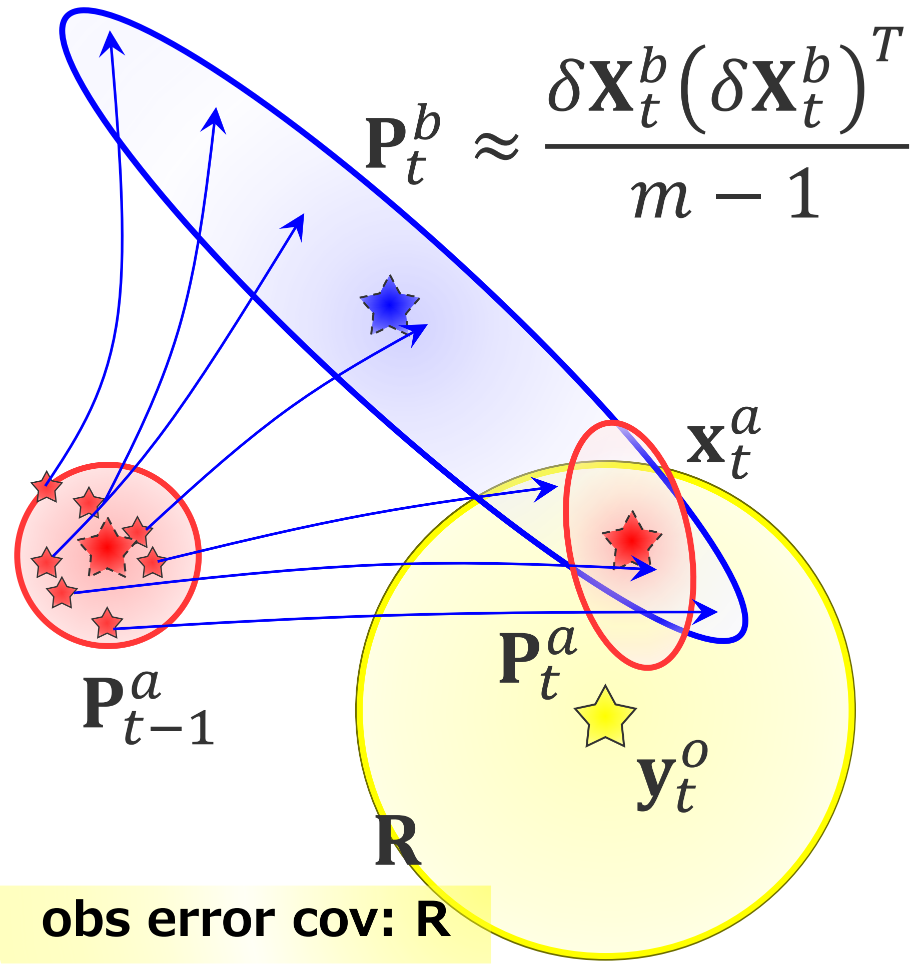
Kalman Filter



Ensemble Kalman Filter



Ensemble Forecasts

Analysis Ensemble

$$\mathbf{X}_{t-1}^{a} = \left[\mathbf{x}_{t-1}^{a(1)}, \mathbf{x}_{t-1}^{a(2)}, \ldots, \mathbf{x}_{t-1}^{a(m)}\right]$$

Ensemble Forecasts

$$\mathbf{X}_{t}^{b} = \left[\mathbf{x}_{t}^{b(1)}, \mathbf{x}_{t}^{b(2)}, \ldots, \mathbf{x}_{t}^{b(m)}\right] \qquad where\ \mathbf{x}_{t}^{b(i)} = M\left(\mathbf{x}_{t-1}^{a(i)}\right)$$

for i = 1,…,m

Ensemble Mean

$$\bar{\mathbf{x}} \equiv \sum_{i=1}^{m} \mathbf{x}^{(i)} / m$$

Ensemble Perturbation δ represents ensemble perturbation

$$\delta\mathbf{X}_{t}^{b} = \left[\mathbf{x}_{t}^{b(1)} - \bar{\mathbf{x}}_{t}^{b}, \mathbf{x}_{t}^{b(2)} - \bar{\mathbf{x}}_{t}^{b}, \ldots, \mathbf{x}_{t}^{b(m)} - \bar{\mathbf{x}}_{t}^{b}\right]$$

$$= \left[\delta\mathbf{x}_{t}^{b(1)}, \delta\mathbf{x}_{t}^{b(2)}, \ldots, \delta\mathbf{x}_{t}^{b(m)}\right]$$

$$\mathbf{Z}_{t}^{b} = \delta\mathbf{X}_{t}^{b} / \sqrt{m-1}$$

Approximation of $\mathbf{P}^b$

Error Propagation in Ensemble Forecasts

$$\overline{M(\mathbf{x}_{t-1}^a)} = \frac{1}{m}\sum_{i=1}^{m} M\left(\mathbf{x}_{t-1}^{a(i)}\right)$$

$$\delta\mathbf{x}_t^{b(i)} = M\left(\mathbf{x}_{t-1}^{a(i)}\right) - \overline{M(\mathbf{x}_{t-1}^a)}$$

$$\approx \left[M(\bar{\mathbf{x}}_{t-1}^a) + \mathbf{M}\delta\mathbf{x}_{t-1}^{a(i)}\right] - [M(\bar{\mathbf{x}}_{t-1}^a) + \langle\mathbf{M}\delta\mathbf{x}_{t-1}^a\rangle]$$

$$= \mathbf{M}\delta\mathbf{x}_{t-1}^{a(i)}$$

$$\frac{1}{m-1}\delta\mathbf{X}_t^b\left(\delta\mathbf{X}_t^b\right)^T$$

$$\approx \frac{1}{m-1}\mathbf{M}\delta\mathbf{X}_{t-1}^a(\mathbf{M}\delta\mathbf{X}_{t-1}^a)^T = \mathbf{M}\mathbf{P}_{t-1}^a\mathbf{M}^T = \mathbf{P}_t^b$$

Ensemble forecasts can be used for approximating propagation of error covariance !

why *m*-1?

- Pls. study discussions on “unbiased variance”

Ensemble Kalman Filter

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Ensemble Perturbation in Observation Space

$$\mathbf{Y}_t^b \equiv \mathbf{H}\mathbf{Z}_t^b \approx \left[H(\mathbf{X}_t^b) - \overline{H(\mathbf{X}_t^b)} \cdot \mathbf{1}\right] / \sqrt{m-1}$$

i.e. no **H** is needed

Error Covariance Approximation

$$\mathbf{P}_t^b \approx \mathbf{Z}_t^b (\mathbf{Z}_t^b)^T = \delta\mathbf{X}_t^b (\delta\mathbf{X}_t^b)^T / (m-1) \qquad \mathbf{H}\mathbf{P}_t^b\mathbf{H}^T \approx \mathbf{Y}_t^b (\mathbf{Y}_t^b)^T$$

Kalman Gain

$$\mathbf{K}_t = \mathbf{P}_t^b \mathbf{H}^T \left[\mathbf{H}\mathbf{P}_t^b\mathbf{H}^T + \mathbf{R}\right]^{-1}$$

$$\approx \mathbf{Z}_t^b (\mathbf{Y}_t^b)^T \left[\underbrace{\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}}_{p \times p}\right]^{-1} = \mathbf{Z}_t^b \left[\underbrace{\mathbf{I} + (\mathbf{Y}_t^b)^T \mathbf{R}^{-1} \mathbf{Y}_t^b}_{m \times m}\right]^{-1} (\mathbf{Y}_t^b)^T \mathbf{R}^{-1}$$

We can choose inverse computation depending on *p* and *m*.

※ usually **R** is diagonal (i.e., no obs error corr.)

$$\mathbf{Z}_t^b (\mathbf{Y}_t^b)^T \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

$$= \mathbf{Z}_t^b \left[\mathbf{I} + (\mathbf{Y}_t^b)^T \mathbf{R}^{-1} \mathbf{Y}_t^b\right]^{-1} \left[\mathbf{I} + (\mathbf{Y}_t^b)^T \mathbf{R}^{-1} \mathbf{Y}_t^b\right] (\mathbf{Y}_t^b)^T \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

$$= \mathbf{Z}_t^b \left[\mathbf{I} + (\mathbf{Y}_t^b)^T \mathbf{R}^{-1} \mathbf{Y}_t^b\right]^{-1} (\mathbf{Y}_t^b)^T \left[\mathbf{I} + \mathbf{R}^{-1} \mathbf{Y}_t^b (\mathbf{Y}_t^b)^T\right] \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

$$= \mathbf{Z}_t^b \left[\mathbf{I} + (\mathbf{Y}_t^b)^T \mathbf{R}^{-1} \mathbf{Y}_t^b\right]^{-1} (\mathbf{Y}_t^b)^T \mathbf{R}^{-1} \left[\mathbf{R} + \mathbf{Y}_t^b (\mathbf{Y}_t^b)^T\right] \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

KF

Prediction (state)

$$\mathbf{x}_t^b = M(\mathbf{x}_{t-1}^a)$$

Prediction of Error Cov. (explicitly)

$$\mathbf{P}_t^b = \mathbf{M}\mathbf{P}_{t-1}^a\mathbf{M}^T(+\mathbf{Q})$$

Kalman Gain

$$\mathbf{K}_t = \mathbf{P}_t^b\mathbf{H}^T\left[\mathbf{H}\mathbf{P}_t^b\mathbf{H}^T + \mathbf{R}\right]^{-1}$$

Analysis (state)

$$\mathbf{x}_t^a = \mathbf{x}_t^b + \mathbf{K}_t(\mathbf{y}_t^o - H(\mathbf{x}_t^b))$$

Analysis Error Covariance

$$\mathbf{P}_t^a = [\mathbf{I} - \mathbf{K}_t\mathbf{H}]\mathbf{P}_t^b$$

EnKF

Ensemble Prediction (state)

$$\mathbf{x}_t^{b(i)} = M\left(\mathbf{x}_{t-1}^{a(i)}\right) \quad \text{for } i = 1,\ldots,m$$

Prediction of Error Covariance (implicitly)

$$\mathbf{P}_t^b \approx \mathbf{Z}_t^b\left(\mathbf{Z}_t^b\right)^T$$

Kalman Gain

$$\mathbf{K}_t = \mathbf{Z}_t^b(\mathbf{Y}_t^b)^T\left[\mathbf{Y}_t^b(\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

$$= \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}$$

Analysis (state)

$$\mathbf{x}_t^a = \mathbf{x}_t^b + \mathbf{K}_t(\mathbf{y}_t^o - H(\mathbf{x}_t^b))$$

Analysis Error Covariance

(1) Stochastic: PO method
(2) Deterministic: Square Root Filter (SRF)
(e.g., serial EnSRF, EAKF, LETKF)

PO Method (stochastic)

Analysis of Ensemble

$\boldsymbol{\varepsilon}_t^o \sim N(0, \mathbf{R})$ randomly drawn perturbation ➔ perturbed observation

$$\mathbf{x}_t^{a(i)} = \mathbf{x}_t^{b(i)} + \mathbf{K}_t(\mathbf{y}_t^o + \boldsymbol{\varepsilon}_t^o - H(\mathbf{x}_t^{b(i)}))$$

Why do we need perturbation?

if w/o perturbation (to take ave. from both sides)

$$\delta\mathbf{x}_t^{a(i)} \approx \delta\mathbf{x}_t^{b(i)} - \mathbf{K}_t\mathbf{H}\delta\mathbf{x}_t^{b(i)}$$

$$\Leftrightarrow \delta\mathbf{X}_t^a \approx (\mathbf{I} - \mathbf{K}_t\mathbf{H})\delta\mathbf{X}_t^b$$

$$\mathbf{P}_t^a \approx \delta\mathbf{X}_t^a(\delta\mathbf{X}_t^a)^T/(m-1)$$

$$= (\mathbf{I} - \mathbf{K}_t\mathbf{H})\mathbf{P}_t^b(\mathbf{I} - \mathbf{K}_t\mathbf{H})^T$$

$$\mathbf{K}_t\mathbf{R}\mathbf{K}_t^T = \mathbf{K}_t\langle\boldsymbol{\varepsilon}_t^o(\boldsymbol{\varepsilon}_t^o)^T\rangle\mathbf{K}_t^T$$

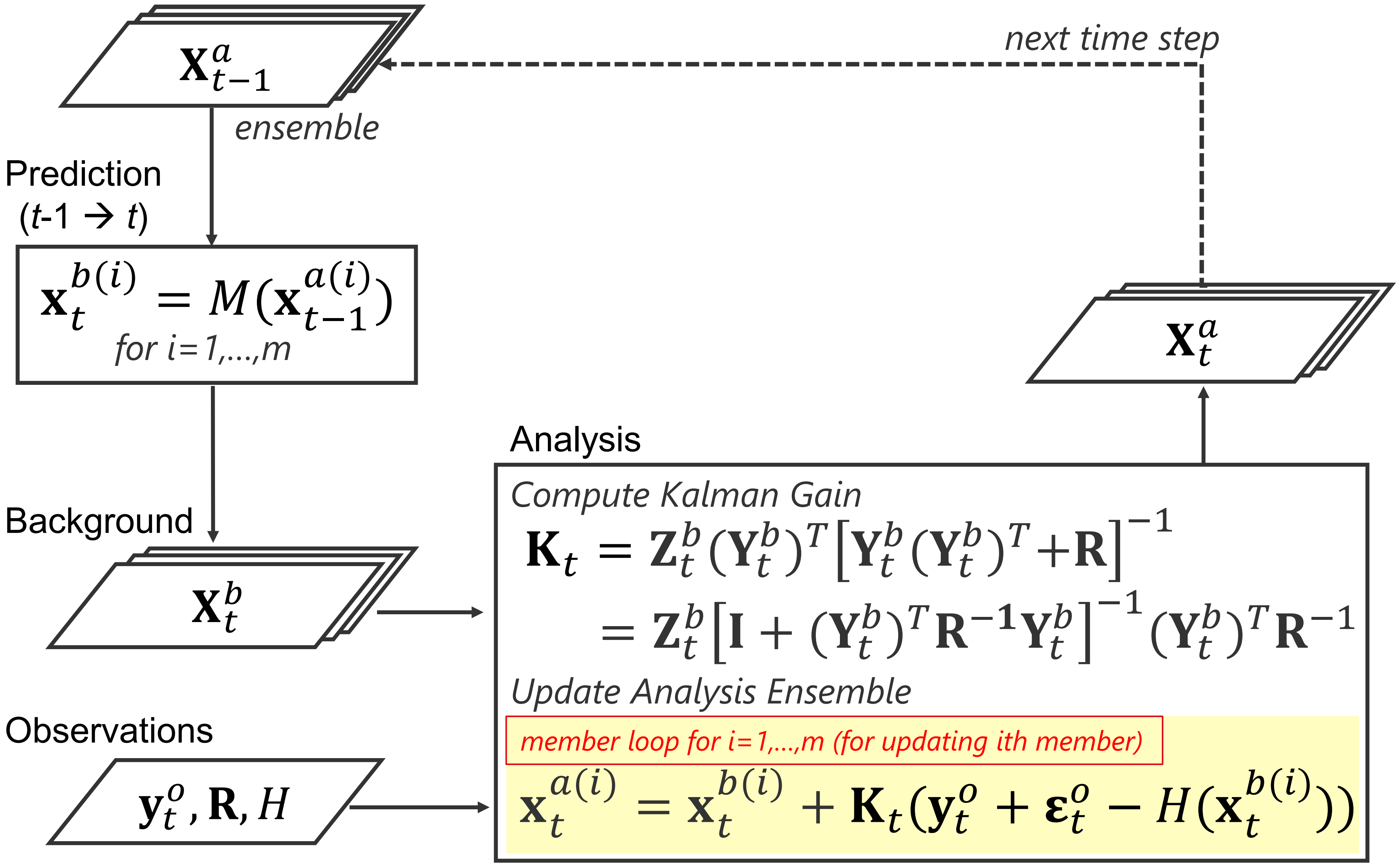
Analysis error covariance is underestimated if without perturbation!

Burgers et al. (1998)

Analysis error covariance should be (cf. 4th lecture)

$$\mathbf{P}_t^a = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_t^b(\mathbf{I} - \mathbf{K}\mathbf{H})^T + \mathbf{K}\mathbf{R}\mathbf{K}^T$$

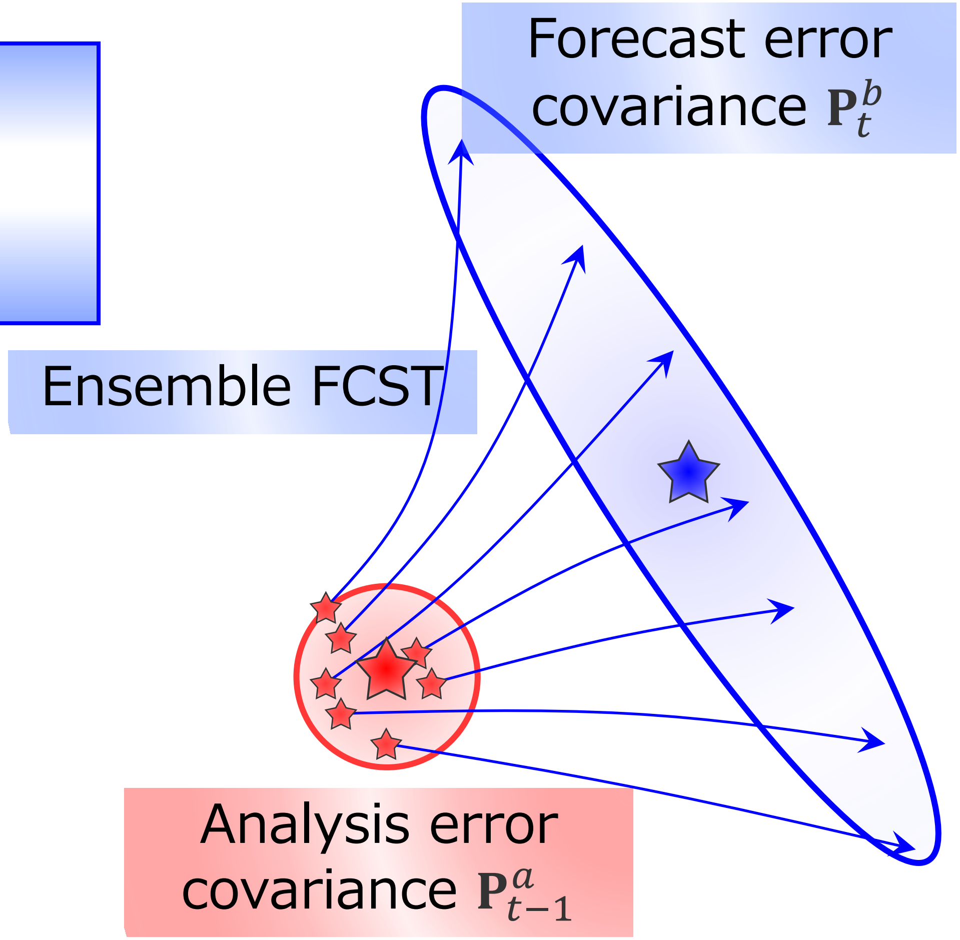
EnKF (PO) Algorithm



Ensemble Kalman Filter

$$\mathbf{X}_t^b = M(\mathbf{X}_{t-1}^a)$$

$$\mathbf{P}_t^b = \mathbf{Z}_t^b \left(\mathbf{Z}_t^b\right)^T$$



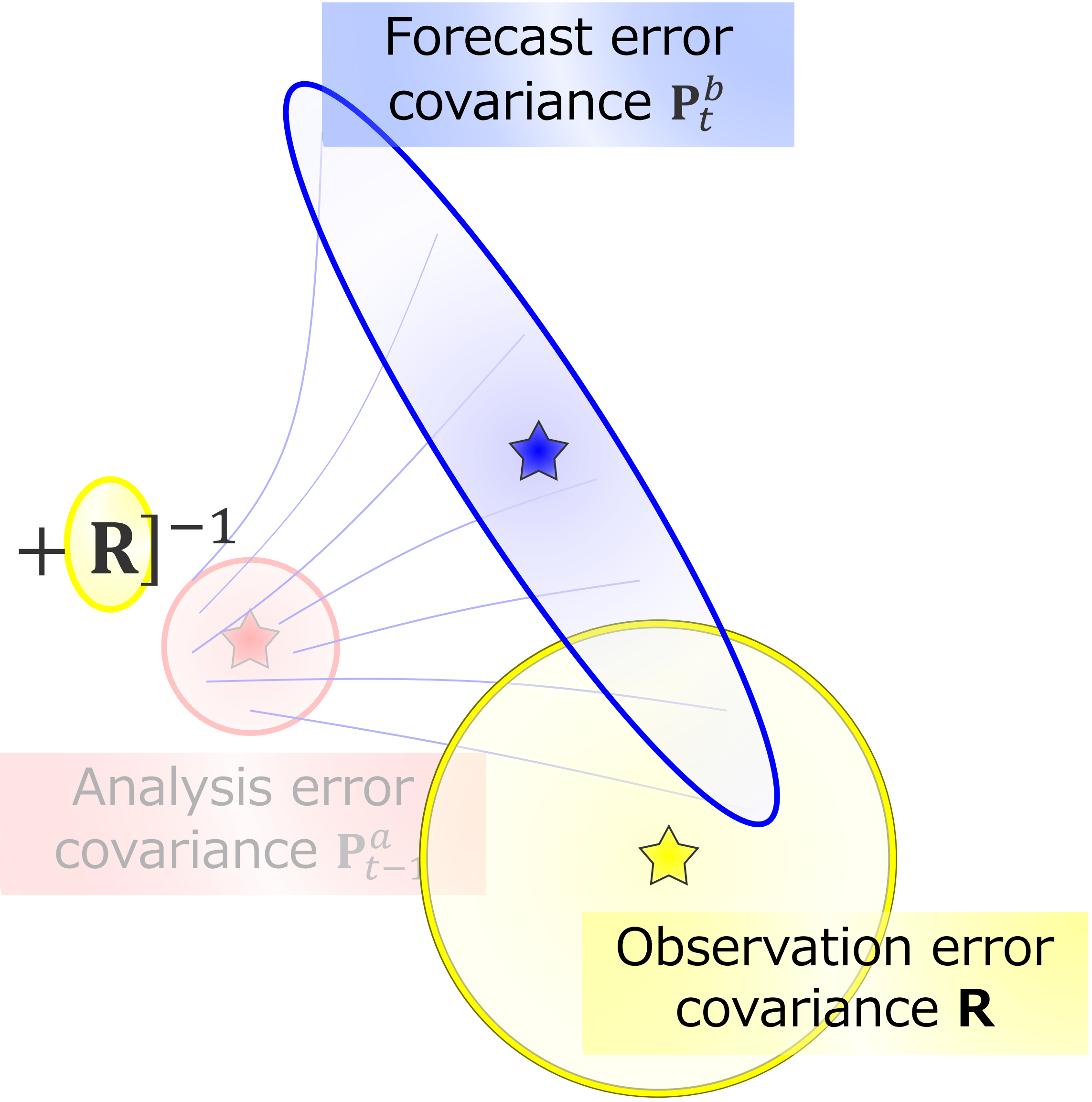
$$\mathbf{Z}_t^b = \delta \mathbf{X}_t^b / \sqrt{m-1}$$

Ensemble Kalman Filter

$$\mathbf{X}_t^b = M(\mathbf{X}_{t-1}^a)$$

$$\mathbf{P}_t^b = \mathbf{Z}_t^b(\mathbf{Z}_t^b)^T$$

$$\mathbf{K}_t = \mathbf{P}_t^b\mathbf{H}^T[\mathbf{H}\mathbf{P}_t^b\mathbf{H}^T + \mathbf{R}]^{-1}$$



Ensemble Kalman Filter

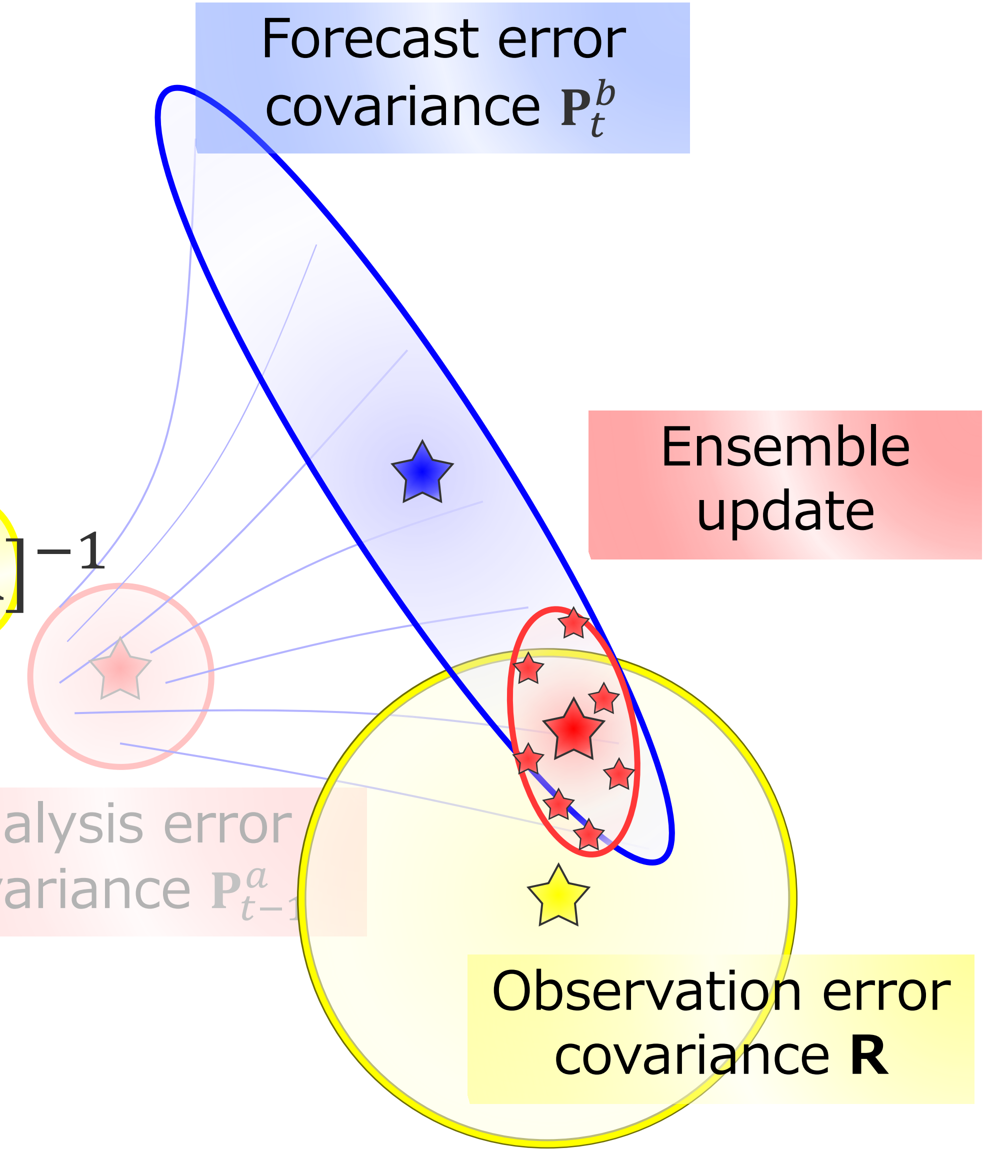
$$\mathbf{X}_t^b = M(\mathbf{X}_{t-1}^a)$$

$$\mathbf{P}_t^b = \mathbf{Z}_t^b(\mathbf{Z}_t^b)^T$$

$$\mathbf{K}_t = \mathbf{P}_t^b \mathbf{H}^T [\mathbf{H}\mathbf{P}_t^b \mathbf{H}^T + \mathbf{R}]^{-1}$$

$$\mathbf{X}_t^a = \mathbf{X}_t^b + \mathbf{K}(\mathbf{y}_t^o - \mathbf{H}\mathbf{X}_t^b)$$

$$\mathbf{P}_t^a = \mathbf{Z}_t^a(\mathbf{Z}_t^a)^\mathrm{T}$$



Ensemble Kalman Filter

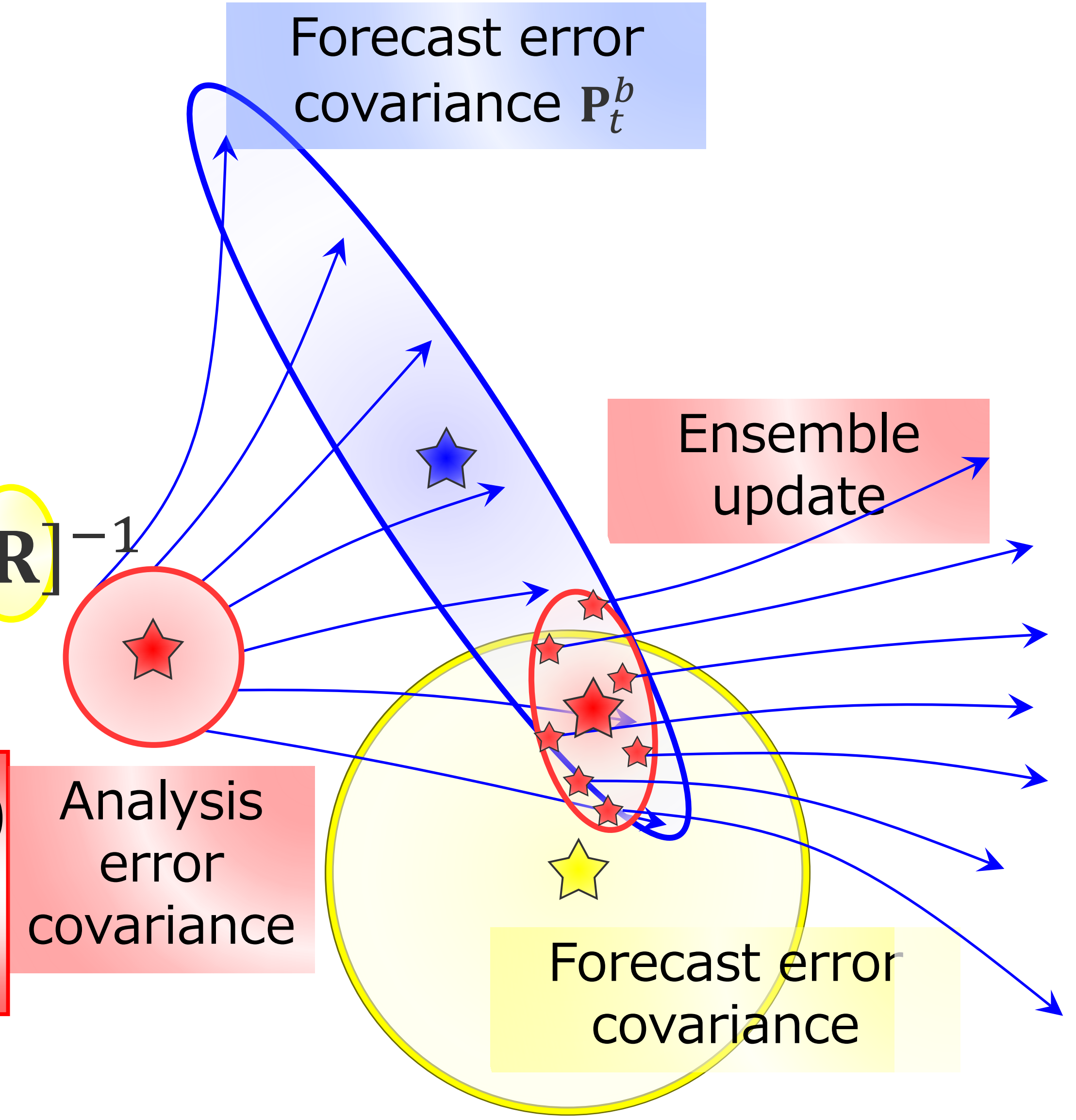
$$\mathbf{X}_t^b = M(\mathbf{X}_{t-1}^a)$$

$$\mathbf{P}_t^b = \delta\mathbf{Z}_t^b\left(\delta\mathbf{Z}_t^b\right)^T$$

$$\mathbf{K}_t = \mathbf{P}_t^b\mathbf{H}^T[\mathbf{H}\mathbf{P}_t^b\mathbf{H}^T + \mathbf{R}]^{-1}$$

$$\mathbf{X}_t^a = \mathbf{X}_t^b + \mathbf{K}(\mathbf{y}_t^o - \mathbf{H}\mathbf{X}_t^b)$$

$$\mathbf{P}_t^a = \mathbf{Z}_t^a(\mathbf{Z}_t^a)^{\mathrm{T}}$$



Basic Task 5

Basic Task 5

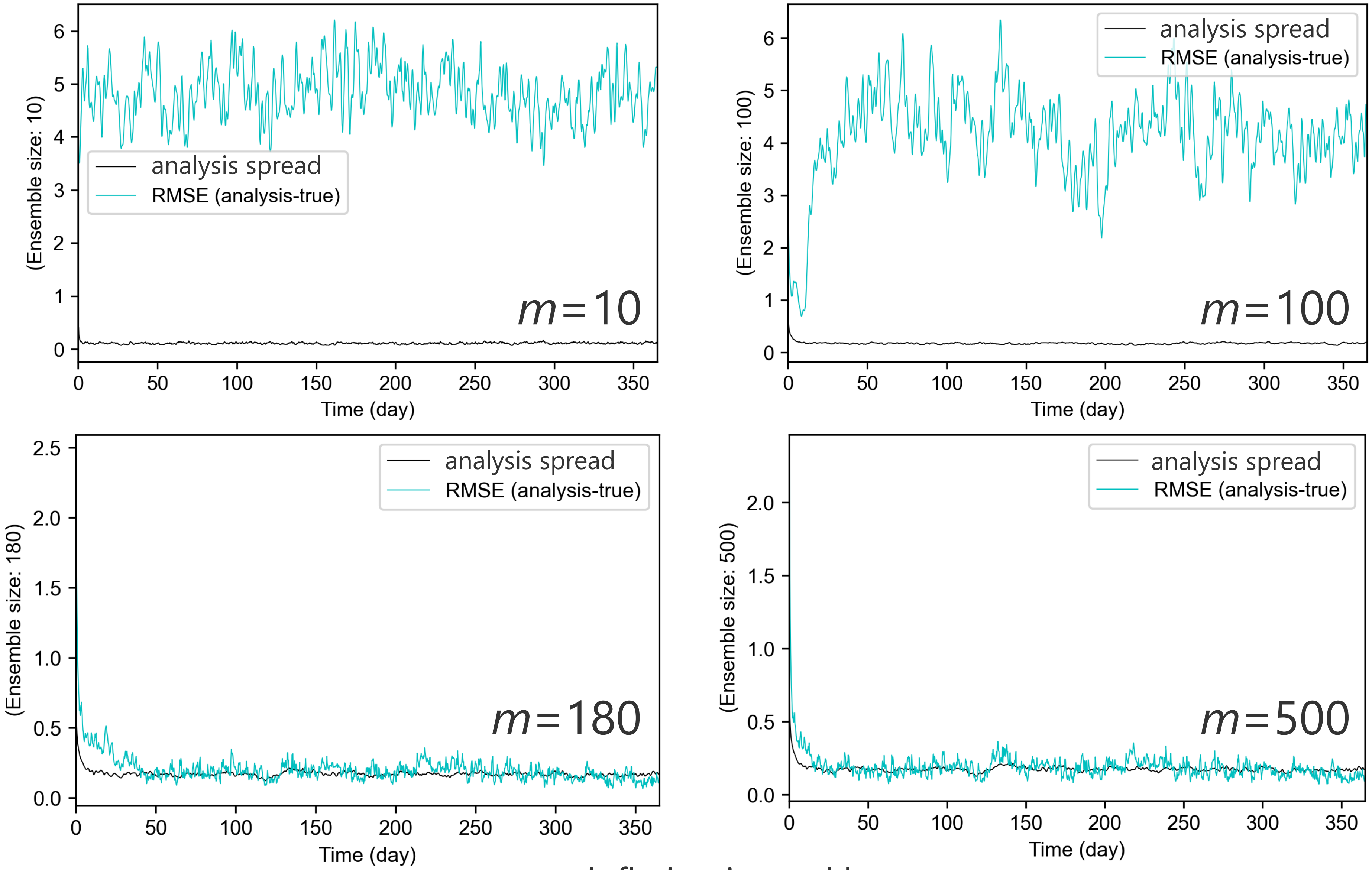
6．EnKF を実装し、KF と比較する。Whitaker and Hamill (2002)による Serial EnSRF, Bishop et al. (2001)による ETKF、Hunt et al. (2007)による LETKF、PO 法などの解法がある。2 つ以上実装すること。

ヒント）気象分野の EnKF では、上述の手法が良く用いられている。カナダでは PO 法、米国気象局では Serial EnSRF、ドイツ・日本では LETKF など。小槻研で研究を進める場合、LETKF を用いた研究をしていくことが想定されるため、LETKF の実装には取り組んで欲しい。

6. Implement EnKF and compare with KF. There are solutions such as Serial EnSRF by Whitaker and Hamill (2002), ETKF by Bishop et al. (2001), LETKF and PO method by Hunt et al. (2007). Implement at least two or more.

Hint) The above methods are often used in EnKF in the meteorological field. PO method in Canada, Serial EnSRF in the US Meteorological Bureau, LETKF in Germany and Japan, etc. When proceeding with research at Kotsuki Lab, it is expected that research using LETKF will be carried out, so I would like you to work on the implementation of LETKF at least.

EnKF (PO) w/o Localization



no inflation is used here

Treatments

(1) Perturbed Observations

Random number should be different for each member and obs

$$\mathbf{x}_t^{a(i)} = \mathbf{x}_t^{b(i)} + \mathbf{K}_t(\mathbf{y}_t^o + \boldsymbol{\varepsilon}_t^{o(i)} - H(\mathbf{x}_t^{b(i)})) \qquad \textit{for } i=1,\ldots,m$$

this error should be modified so that $\sum_{i=1}^{m} \boldsymbol{\varepsilon}_t^{o(i)} = 0$

$$\Leftrightarrow \bar{\mathbf{x}}_t^a = \bar{\mathbf{x}}_t^b + \mathbf{K}_t(\mathbf{y}_t^o - \overline{H(\mathbf{x}_t^b)})$$

Specifically, generate random numbers $\boldsymbol{\varepsilon}_t'^{o(i)}$ $for\ i = 1,,,m$
and compute their average $\bar{\boldsymbol{\varepsilon}}_t^o = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{\varepsilon}_t'^{o(i)}$.
Then, perturb with $\boldsymbol{\varepsilon}_t^{o(i)} = \boldsymbol{\varepsilon}_t'^{o(i)} - \bar{\boldsymbol{\varepsilon}}_t^o$ that satisfies $\sum_{i=1}^{m} \boldsymbol{\varepsilon}_t^{o(i)} = 0$

(2) Variance Inflation

$$\mathbf{P}_{inf}^b = (1+\delta)^2 \cdot \mathbf{P}^b$$

$$\Leftrightarrow \delta\mathbf{X}_{inf}^b = (1+\delta) \cdot \delta\mathbf{X}^b$$

(3) Localization

Assimilating surrounding local obs. to update centered grid x1 (★)

- to limit impacts of obs far from analysis grid points
  for erroneous error "co"variance due to sampling errors
- several localizations have been proposed
  - **K** localization (in PO or serial EnKF
  - **R** localization (in LETKF)
  - **B** localization (usually complex for high-dim modes)

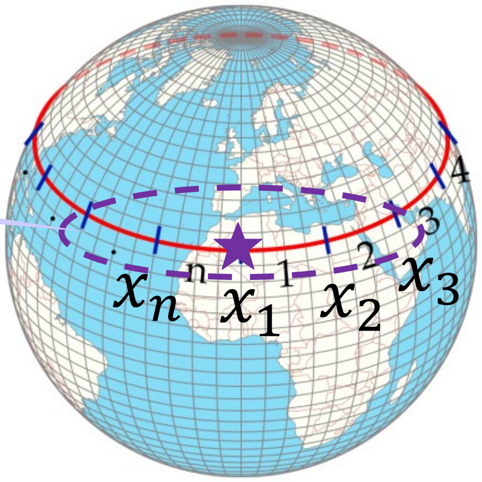


Figure 1.3: Example of a latitude circle of the earth, divided into n equal sized sectors.

Kekem and Leendert (2018)

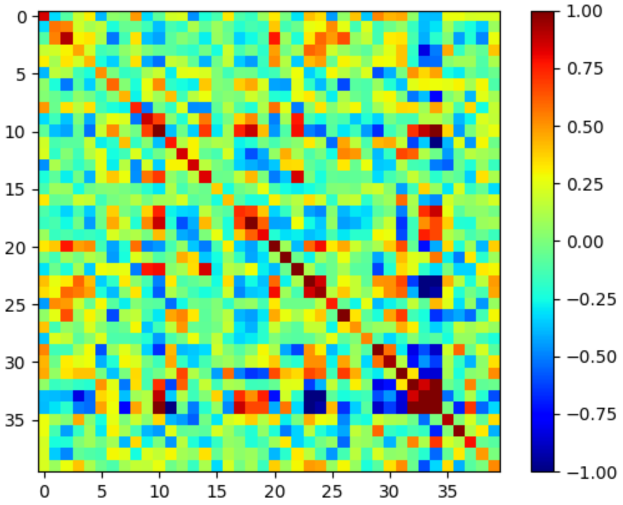
Covariance localization (EnKF)

Empirical treatment for
(1) reducing sampling noise
(2) increasing the rank

$$\mathbf{P}^b \rightarrow \mathbf{L} \circ \mathbf{P}^b$$

$\circ$: Schur product

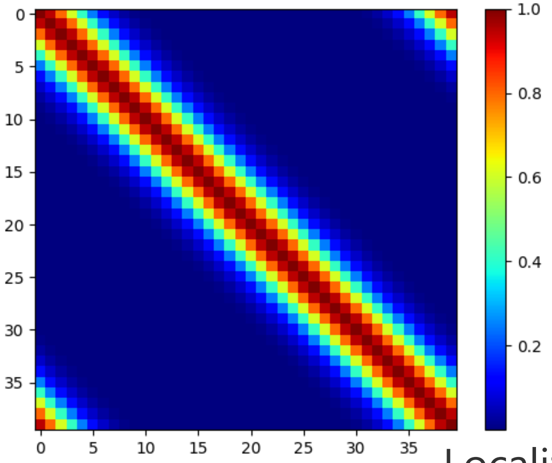
$$\mathbf{P}^b \approx \frac{1}{m-1} \delta\mathbf{X}^b \left(\delta\mathbf{X}^b\right)^{\mathrm{T}}$$



Sampled error covariance (ensemble approximation)

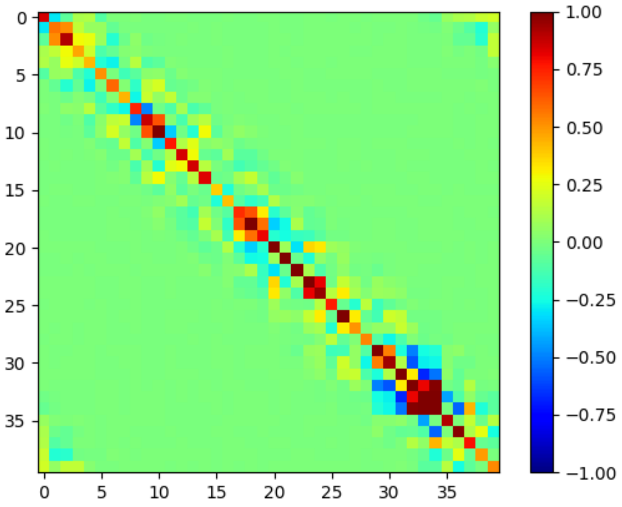
localization

$$\mathbf{L}$$



Localization Function

$$\mathbf{L} \circ \mathbf{P}^b$$



Error Cov. w/ Localization

Localization Function

Gaspari Cohn Function

$$r = \frac{d}{\sqrt{10/3}\,\sigma}$$

tuning parameter

d: distance b/w grids
σ: localization length scale

$$L(r) = \begin{cases} 1 - \frac{1}{4}r^5 + \frac{1}{2}r^4 + \frac{5}{8}r^3 - \frac{5}{3}r^2 & (r \le 1) \\ \frac{1}{12}r^5 - \frac{1}{2}r^4 + \frac{5}{8}r^3 + \frac{5}{3}r^2 - 5r + 4 - \frac{2}{3}r^{-1} & (1 < r \le 2) \\ 0 & (2 < r) \end{cases}$$

Gaspari and Cohn (1999)

usually used in PO and serial EnSRF

Gaussian Function

$$L(d) = \begin{cases} \exp(-\frac{d^2}{2\sigma^2}) & d < 2\sqrt{10/3}\sigma \\ 0 & else \end{cases}$$

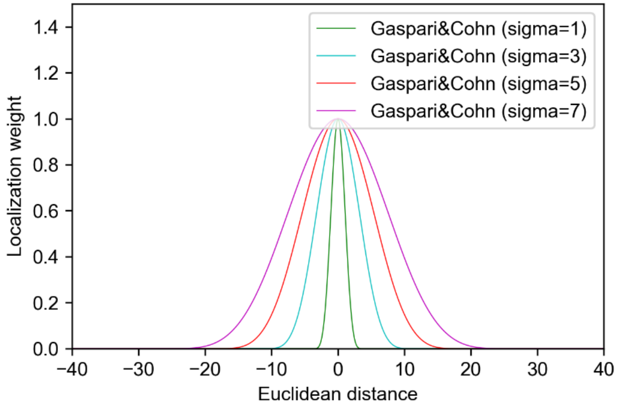
tuning parameter

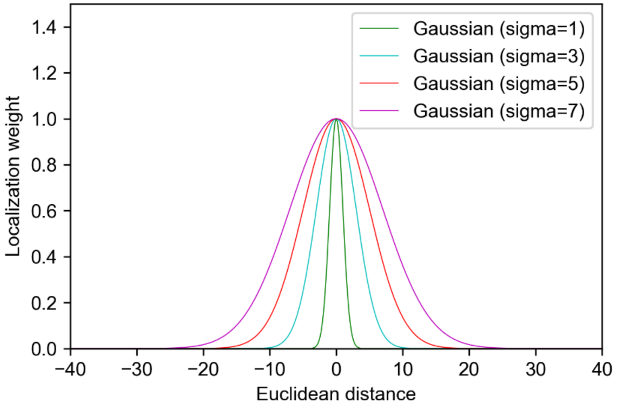
d: distance b/w grids
σ: localization length scale

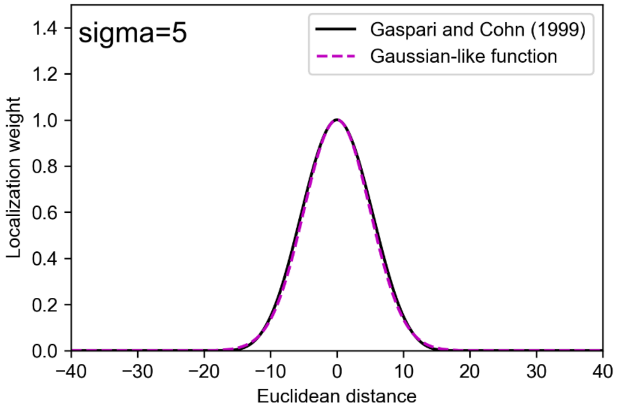
usually used in LETKF

Localization Function

Localization Function

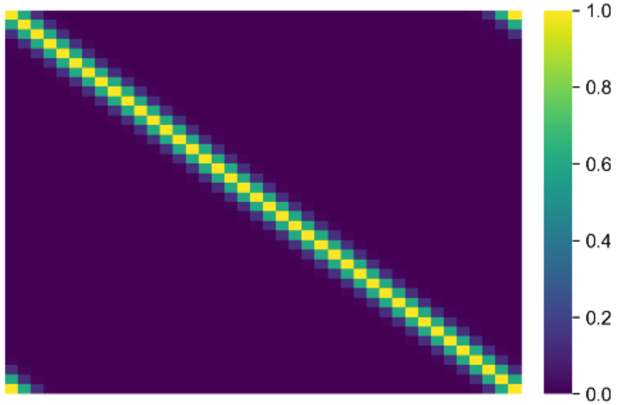




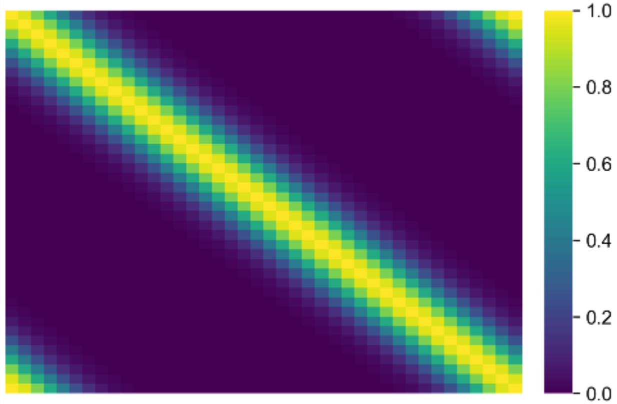


Localization Matrix (Gaussian Function)

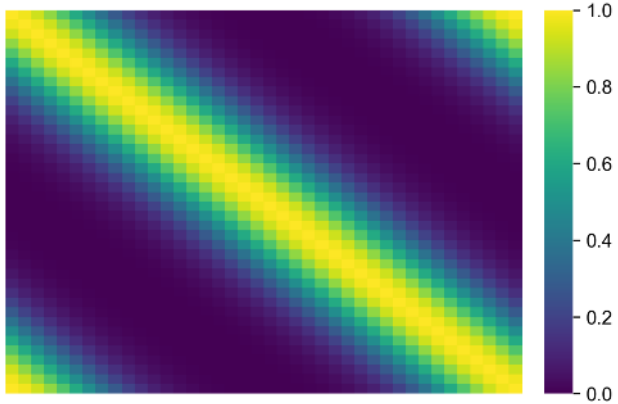
$\sigma = 1$



$\sigma = 3$



$\sigma = 5$



Localization in PO method

Kalman Gain

$$\mathbf{K}_t = \mathbf{Z}_t^b (\mathbf{Y}_t^b)^T \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

$$\mathbf{P}_t^b \mathbf{H}^T = \mathbf{L} \circ \mathbf{Z}_t^b (\mathbf{Y}_t^b)^T$$

$$\mathbf{H}\mathbf{P}_t^b \mathbf{H}^T = \mathbf{L} \circ \mathbf{Y}_t^b (\mathbf{Y}_t^b)^T$$

$\mathbf{L}$: Localization Matrix

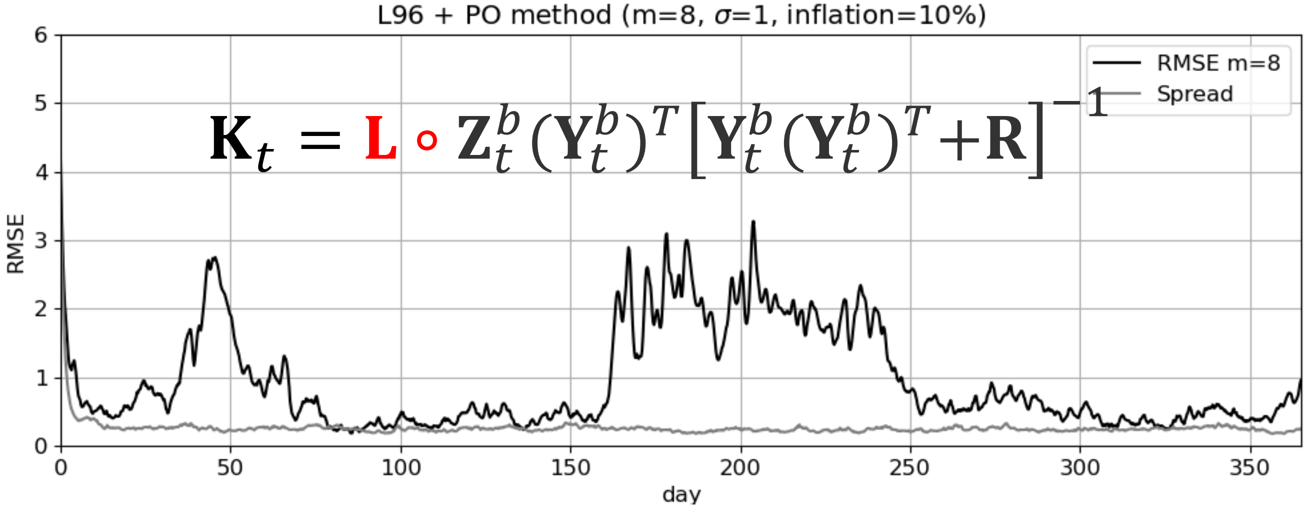
$\circ$: Shur product
also known as Hadamard product
or, element-wise product

Impacts of Localization (PO)

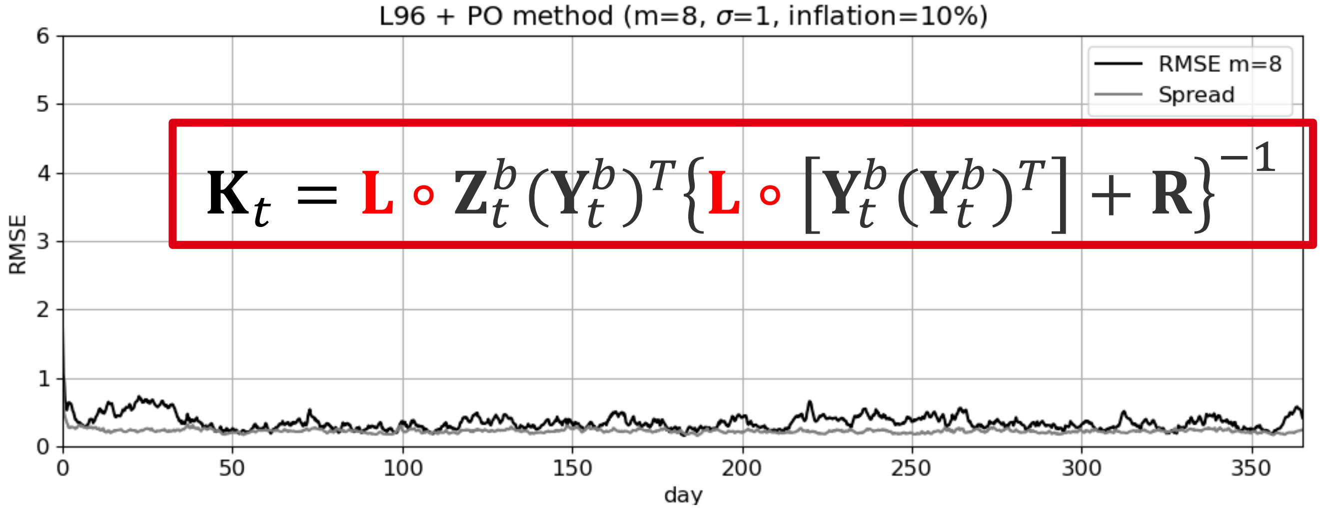
m=8, σ=1, δ=10%

Inflation : $\sqrt{\delta} \circ \mathbf{Z}_t^b$

$$\mathbf{K}_t = \mathbf{L} \circ \mathbf{Z}_t^b (\mathbf{Y}_t^b)^T \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T + \mathbf{R}\right]^{-1}$$

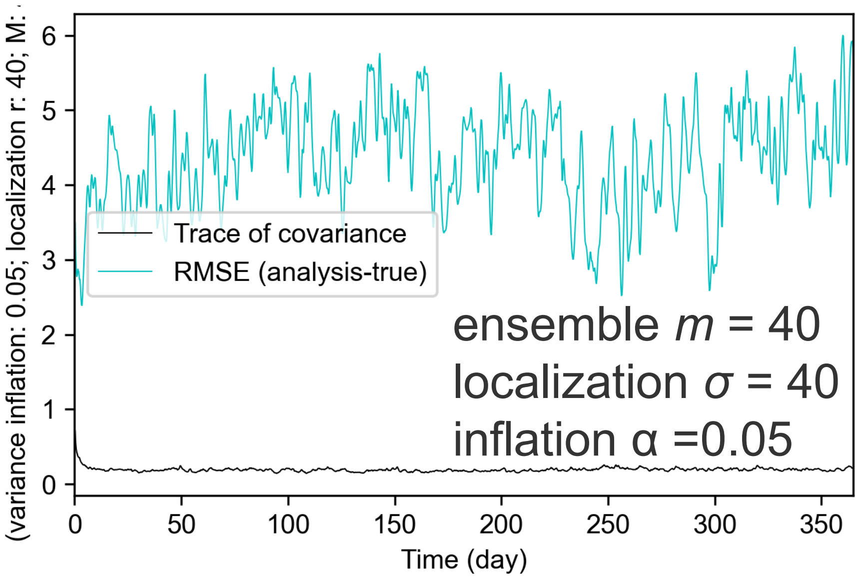


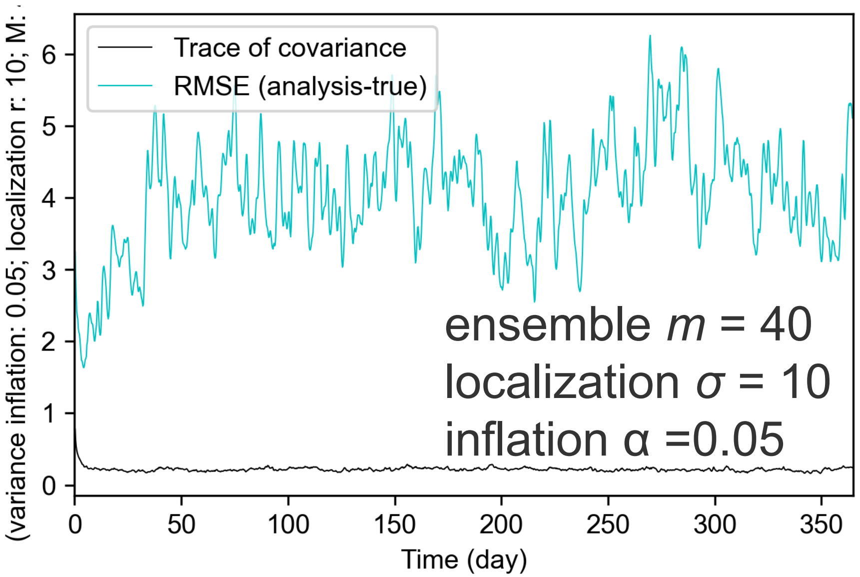
$$\mathbf{K}_t = \mathbf{L} \circ \mathbf{Z}_t^b (\mathbf{Y}_t^b)^T \left\{\mathbf{L} \circ \left[\mathbf{Y}_t^b (\mathbf{Y}_t^b)^T\right] + \mathbf{R}\right\}^{-1}$$

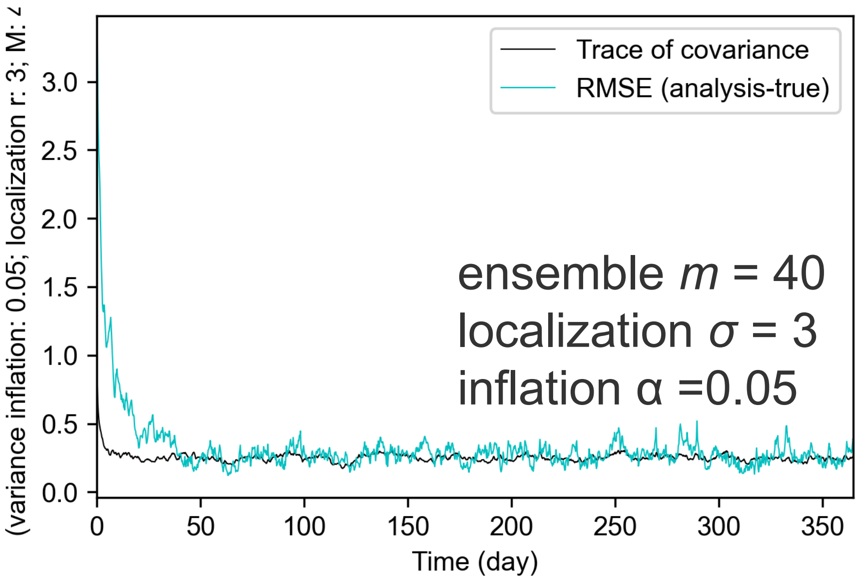


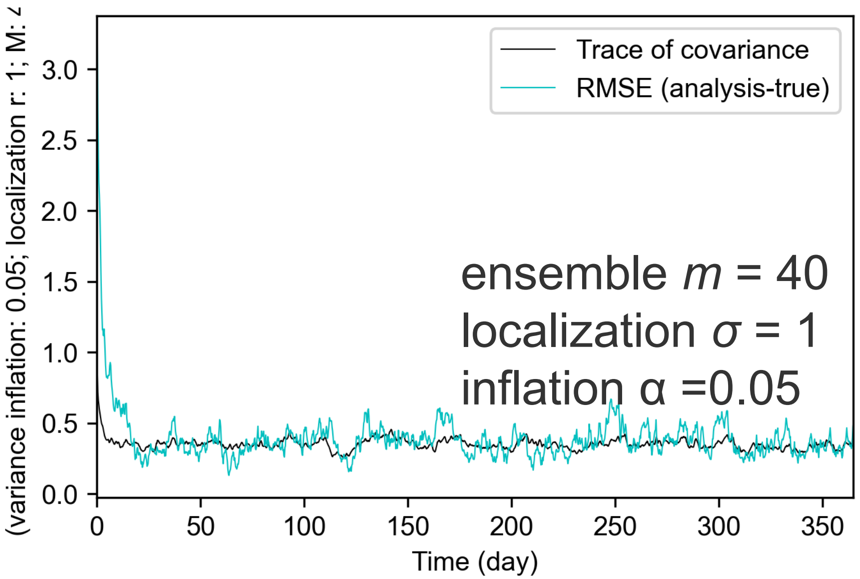
Localization should be applied for all “co”variance

EnKF (PO) w/ Localization









Implementing PO method via ens. transform mtx

Tsuyuki, T., 2024:
A hybrid ensemble Kalman filter to mitigate non-Gaussianity in nonlinear data assimilation.
J. Meteor. Soc. Japan , **102** , 507-524.

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Transform mtx of PO Method (1)

Analysis of Ensemble

Random number should be different for each member and obs

$$\mathbf{x}_t^{a(i)} = \mathbf{x}_t^{b(i)} + \mathbf{K}_t(\mathbf{y}_t^o + \boldsymbol{\varepsilon}_t^{o(i)} - H(\mathbf{x}_t^{b(i)}))$$

for i=1,...,m

this error should be modified so that $\sum_{i=1}^{m} \boldsymbol{\varepsilon}_t^{o(i)} = 0 \Leftrightarrow \bar{\mathbf{x}}_t^a = \bar{\mathbf{x}}_t^b + \mathbf{K}_t(\mathbf{y}_t^o - \overline{H(\mathbf{x}_t^b)})$

Namely, analysis mean $\bar{\mathbf{x}}_t^a$ is equal to that of EnSRFs (e.g. LETKF)
(Strictly speaking, they can be different because of **B** or **R**-localizations)

$$\Leftrightarrow \mathbf{X}_t^a = \mathbf{X}_t^b + \mathbf{K}_t(\mathbf{y}_t^o \cdot \mathbf{1} + \mathbf{E}_t^o - H(\mathbf{X}_t^b))$$

↓ minus analysis mean

Analysis of Ensemble Perturbations

$$\delta\mathbf{X}_t^a \approx \delta\mathbf{X}_t^b + \mathbf{K}_t(\mathbf{E}_t^o - \mathbf{H}\delta\mathbf{X}_t^b)$$

$$\Leftrightarrow \mathbf{Z}_t^a \approx \mathbf{Z}_t^b + \mathbf{K}_t(\mathbf{E}_t^o/\sqrt{m-1} - \mathbf{Y}_t^b)$$

Transform mtx of PO Method (2)

Analysis of Ensemble Perturbations (PO)

$$\mathbf{Z}_t^a \approx \mathbf{Z}_t^b + \mathbf{K}_t(\mathbf{E}_t^o/\sqrt{m-1} - \mathbf{Y}_t^b)$$

Kalman gain $\mathbf{K}_t = \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}$

$$\Leftrightarrow \mathbf{Z}_t^a \approx \mathbf{Z}_t^b\left(\mathbf{I} - \left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right) + \mathbf{K}_t(\mathbf{E}_t^o/\sqrt{m-1})$$

$$= \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1} + \mathbf{K}_t(\mathbf{E}_t^o/\sqrt{m-1})$$

Analysis of Ensemble Perturbations (ETKF)

$$\mathbf{Z}_t^a = \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1/2}$$

analysis ensemble can be explained by ensemble transform matrix!!!

$$\mathbf{I} - \left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b$$

$$= \left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right] - \left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b$$

$$= \left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}$$

Hybrid (LETKF-PO)

Analysis Mean

should be equivalent for LETKF and PO $\bar{\mathbf{x}}_t^a = \bar{\mathbf{x}}_t^b + \mathbf{K}_t(\mathbf{y}_t^o - \overline{H(\mathbf{x}_t^b)})$

Analysis Ensemble

PO
$$\mathbf{Z}_{t,PO}^a = \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1} + \mathbf{K}_t(\mathbf{E}_t^o/\sqrt{m-1})$$
$$= \mathbf{Z}_t^b\mathbf{C}\mathbf{D}^{-1}\mathbf{C}^T + \mathbf{Z}_t^b\mathbf{C}\mathbf{D}^{-1}\mathbf{C}^T(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}(\mathbf{E}_t^o/\sqrt{m-1})$$
$$= \mathbf{Z}_t^b\mathbf{C}\mathbf{D}^{-1}\mathbf{C}^T(\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}(\mathbf{E}_t^o/\sqrt{m-1}))$$

ETKF
$$\mathbf{Z}_{t,ETKF}^a = \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1/2}$$
$$= \mathbf{Z}_t^b\mathbf{C}\mathbf{D}^{-1/2}\mathbf{C}^T$$

(1) no inflation is necessary (unlike R)
(2) the same random numbers should be used for the same obs for different local analysis (to produce spatially smooth AN)

Hybrid
$$\mathbf{Z}_{t,hybrid}^a = (1-w)\mathbf{Z}_{t,ETKF}^a + w\mathbf{Z}_{t,PO}^a$$

Eigenvalue Decomposition $(\widetilde{\mathbf{P}}^a)^{-1} = \mathbf{I} + (\mathbf{Y}^b)^T\mathbf{R}^{-1}\mathbf{Y}^b = \mathbf{C}\mathbf{D}\mathbf{C}^T$

Kalman Gain $\mathbf{K}_t = \mathbf{Z}_t^b\left[\mathbf{I} + (\mathbf{Y}_t^b)^T\mathbf{R}^{-1}\mathbf{Y}_t^b\right]^{-1}(\mathbf{Y}_t^b)^T\mathbf{R}^{-1}$

Hybrid (LETKF-PO), memo

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Analysis Ensemble (PO)

$$\mathbf{Z}^a_{t,PO} = \mathbf{Z}^b_t\left[\mathbf{I} + (\mathbf{Y}^b_t)^T\mathbf{R}^{-1}\mathbf{Y}^b_t\right]^{-1}(\mathbf{I} + (\mathbf{Y}^b_t)^T\mathbf{R}^{-1}(\mathbf{E}^o_t/\sqrt{m-1}))$$

$$\delta\mathbf{X}^a_{t,PO} = \delta\mathbf{X}^b_t\left[\mathbf{I} + (\mathbf{Y}^b_t)^T\mathbf{R}^{-1}\mathbf{Y}^b_t\right]^{-1}(\mathbf{I} + (\mathbf{Y}^b_t)^T\mathbf{R}^{-1}(\mathbf{E}^o_t/\sqrt{m-1}))$$

$$\mathbf{Y}^b = \delta\mathbf{Y}^b/\sqrt{m-1}$$

$$\delta\mathbf{X}^a_{t,PO} = \delta\mathbf{X}^b_t\left[(m-1)\,\mathbf{I} + (\delta\mathbf{Y}^b_t)^T\mathbf{R}^{-1}\delta\mathbf{Y}^b_t\right]^{-1}((m-1)\mathbf{I} + (\delta\mathbf{Y}^b_t)^T\mathbf{R}^{-1}\mathbf{E}^o_t)$$

$(\widetilde{\mathbf{P}}^a)$ *of Hunt LETKF*

Analysis Ensemble (LETKF)

$$\delta\mathbf{X}^a_{t,LETKF} = \delta\mathbf{X}^b_t[\mathbf{I} + (\mathbf{Y}^b_t)^T\mathbf{R}^{-1}\mathbf{Y}^b_t]^{-1/2}$$

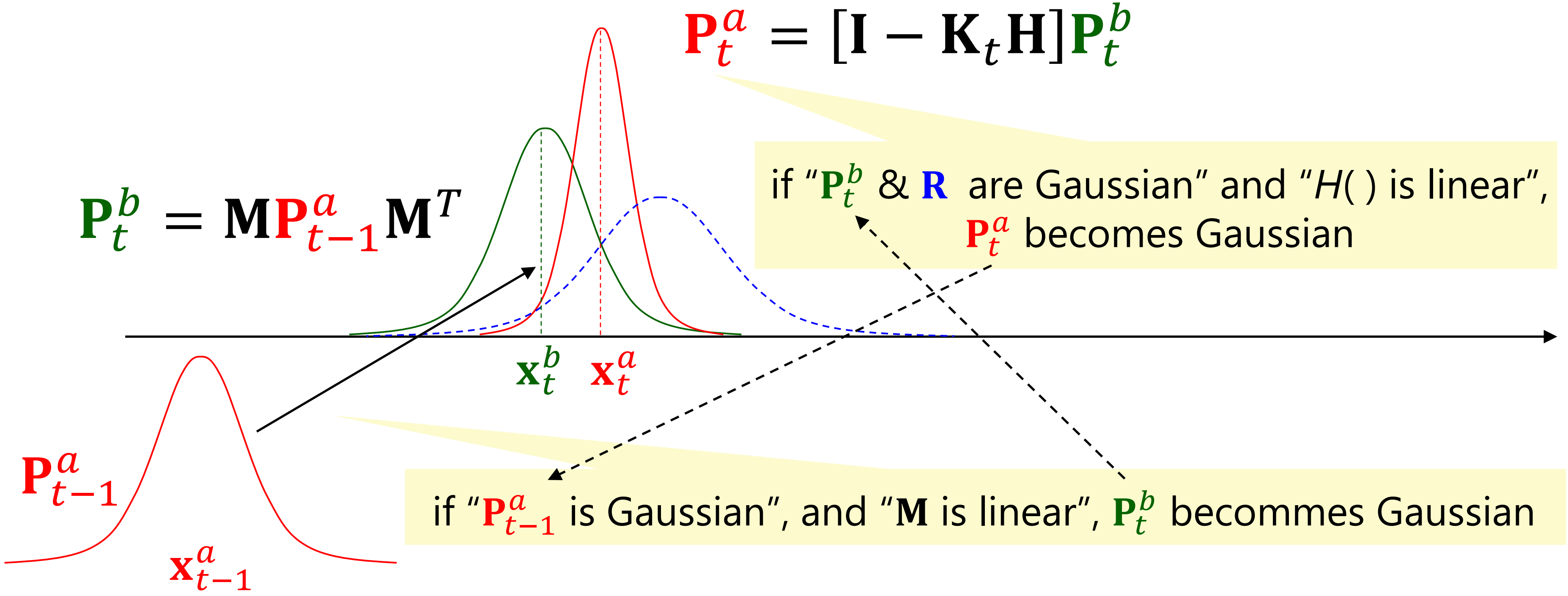
$$\mathbf{Y}^b = \delta\mathbf{Y}^b/\sqrt{m-1}$$

$(\widetilde{\mathbf{P}}^a)^{1/2}$ *of Hunt LETKF*

$$\delta\mathbf{X}^a_{t,LETKF} = \delta\mathbf{X}^b_t\sqrt{m-1}[(m-1)\mathbf{I} + (\delta\mathbf{Y}^b_t)^T\mathbf{R}^{-1}\delta\mathbf{Y}^b_t]^{-1/2}$$

$$\mathbf{Z}^b \equiv \delta\mathbf{X}^b/\sqrt{m-1} \quad \delta\mathbf{Y}^b \equiv \mathbf{H}\delta\mathbf{X}^b \quad \mathbf{Y}^b \equiv \mathbf{H}\mathbf{Z}^b = \delta\mathbf{Y}^b/\sqrt{m-1}$$

Connection to non-Gaussianity



In other words, non-Gaussian $\mathbf{P}_{t-1}^a$ leads to non-Gaussian $\mathbf{P}_t^b$ even if $\mathbf{M}$ is linear.
The non-Gaussian $\mathbf{P}_t^b$ yields non-Gaussian $\mathbf{P}_t^a$ if without any treatments.

Tsuyuki (2024) pointed out that PO can produce more Gaussian $\mathbf{P}_t^a$ because of

$$\mathbf{X}_t^a = \mathbf{X}_t^b + \mathbf{K}_t(\mathbf{y}_t^o \cdot \mathbf{1} - H(\mathbf{X}_t^b)) + \mathbf{K}_t\mathbf{E}_t^o$$

Gaussian perturbation

Thank you for your attention!

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Further information is available at

https://kotsuki-lab.com/