Data Assimilation
- O02. Degrees of Freedom for the Signal -

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Degrees of Freedom for the Signal (DFS)

Preparation: SVD of Observability Matrix

$$\mathbf{G} = \mathbf{R}^{-1/2}\mathbf{H}\mathbf{B}^{1/2} \underset{SVD}{=} \mathbf{E}\boldsymbol{\Gamma}^{1/2}\mathbf{C}^T$$

$\mathbf{E} \in \mathbb{R}^{p\times p}$, $\boldsymbol{\Gamma}^{1/2} \in \mathbb{R}^{p\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times n}$

here $\boldsymbol{\Gamma}^{1/2} = \begin{bmatrix} \boldsymbol{\Gamma}_r^{\frac{1}{2}} & \boldsymbol{O}_{r\times(n-r)} \\ \boldsymbol{O}_{(p-r)\times r} & \boldsymbol{O}_{(p-r)\times(n-r)} \end{bmatrix}$ *when* $r = rank(\mathbf{G})$

$$\mathbf{E}\boldsymbol{\Gamma}^{1/2} = \begin{bmatrix} \mathbf{E}_r\boldsymbol{\Gamma}_r^{1/2} & \boldsymbol{O}_{p\times(n-r)} \end{bmatrix}$$

DFS

$$DFS \equiv trace(\mathbf{S}) \qquad \mathbf{S} = \frac{\partial \mathbf{y}^a}{\partial \mathbf{y}^o} = \frac{\partial}{\partial \mathbf{y}^o}(\mathbf{H}\mathbf{x}^a) = \frac{\partial}{\partial \mathbf{y}^o}\left(\mathbf{H}\mathbf{x}^b + \mathbf{H}\mathbf{K}\left(\mathbf{y}^o - \mathbf{H}\mathbf{x}^b\right)\right) = \mathbf{HK}$$

$$DFS = trace(\mathbf{HBH}^T[\mathbf{HBH}^T + \mathbf{R}]^{-1})$$

$$= trace(\mathbf{HBH}^T\mathbf{R}^{-1/2}\left[\mathbf{R}^{-1/2}\mathbf{HBH}^T\mathbf{R}^{-1/2} + \mathbf{I}\right]^{-1})\mathbf{R}^{-1/2}$$

$$= trace(\mathbf{R}^{-1/2}\mathbf{HBH}^T\mathbf{R}^{-1/2}\left[\mathbf{R}^{-1/2}\mathbf{HBH}^T\mathbf{R}^{-1/2} + \mathbf{I}\right]^{-1})$$

$\mathrm{trace}\,\mathbf{AB} = \mathrm{trace}\,\mathbf{BA}$

$$= trace(\mathbf{GG}^T[\mathbf{GG}^T + \mathbf{I}]^{-1})$$

where $\mathbf{G} = \mathbf{R}^{-1/2}\mathbf{HB}^{1/2}$ observability matrix

$$= \mathbf{E\Gamma}^{1/2}(\mathbf{E\Gamma}^{1/2})^T = \mathbf{E}_r\mathbf{\Gamma}_r\mathbf{E}_r^T$$

$$= trace(\mathbf{E}_r\mathbf{\Gamma}_r\mathbf{E}_r^T[\mathbf{E}_r\mathbf{\Gamma}_r\mathbf{E}_r^T + \mathbf{I}]^{-1})$$

DFS (cont'd)

Using Sherman-Morrison-Woodbury formula

$$(\mathbf{A}+\mathbf{BDC})^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}[\mathbf{D}^{-1}+\mathbf{CA}^{-1}\mathbf{B}]^{-1}\mathbf{CA}^{-1}$$

$$DFS = trace(\mathbf{E}_r\boldsymbol{\Gamma}_r\mathbf{E}_r^T[\mathbf{I}+\mathbf{E}_r\boldsymbol{\Gamma}_r\mathbf{E}_r^T]^{-1})$$

$$= trace(\mathbf{E}_r\boldsymbol{\Gamma}_r\mathbf{E}_r^T\{\mathbf{I}-\mathbf{E}_r[\boldsymbol{\Gamma}_r^{-1}+\mathbf{E}_r^T\mathbf{E}_r]^{-1}\mathbf{E}_r^T\})$$

$$= trace(\mathbf{E}_r\{\boldsymbol{\Gamma}_r\mathbf{E}_r^T-\boldsymbol{\Gamma}_r\cancel{\mathbf{E}_r^T\mathbf{E}_r}[\boldsymbol{\Gamma}_r^{-1}+\mathbf{I}]^{-1}\mathbf{E}_r^T\})$$

$$= trace(\mathbf{E}_r\{\boldsymbol{\Gamma}_r-\boldsymbol{\Gamma}_r[\boldsymbol{\Gamma}_r^{-1}+\mathbf{I}]^{-1}\}\mathbf{E}_r^T)$$

$$= trace(\boldsymbol{\Gamma}_r-\boldsymbol{\Gamma}_r[\boldsymbol{\Gamma}_r^{-1}+\mathbf{I}]^{-1})$$

$$= trace(\boldsymbol{\Gamma}_r[\boldsymbol{\Gamma}_r^{-1}+\mathbf{I}]^{-1}) \qquad = \sum_{i=1}^{r}\frac{\gamma_i}{1+\gamma_i}$$

Thank you for your attention!

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Further information is available at
https://kotsuki-lab.com/

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