

# Data Assimilation

## - 002. Degrees of Freedom for the Signal -

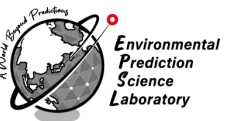
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# Degrees of Freedom for the Signal (DFS)

# Preparation: SVD of Observability Matrix

$$\mathbf{G} = \mathbf{R}^{-1/2} \mathbf{H} \mathbf{B}^{1/2} \underset{\text{SVD}}{=} \mathbf{E} \mathbf{\Gamma}^{1/2} \mathbf{C}^T \quad \mathbf{E} \in \mathbb{R}^{p \times p}, \quad \mathbf{\Gamma}^{1/2} \in \mathbb{R}^{p \times n}, \quad \mathbf{C} \in \mathbb{R}^{n \times n}$$

$$\text{here } \mathbf{\Gamma}^{1/2} = \begin{bmatrix} \mathbf{\Gamma}_r^{1/2} & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(p-r) \times r} & \mathbf{0}_{(p-r) \times (n-r)} \end{bmatrix} \quad \text{when } r = \text{rank}(\mathbf{G})$$

$$\mathbf{E} \mathbf{\Gamma}^{1/2} = \begin{bmatrix} \mathbf{E}_r \mathbf{\Gamma}_r^{1/2} & \mathbf{0}_{p \times (n-r)} \end{bmatrix}$$

$$DFS \equiv \text{trace}(\mathbf{S}) \quad \mathbf{S} = \frac{\partial \mathbf{y}^a}{\partial \mathbf{y}^o} = \frac{\partial}{\partial \mathbf{y}^o} (\mathbf{H}\mathbf{x}^a) = \frac{\partial}{\partial \mathbf{y}^o} (\mathbf{H}\mathbf{x}^b + \mathbf{H}\mathbf{K}(\mathbf{y}^o - \mathbf{H}\mathbf{x}^b)) = \mathbf{H}\mathbf{K}$$

$$DFS = \text{trace}(\mathbf{H}\mathbf{B}\mathbf{H}^T [\mathbf{H}\mathbf{B}\mathbf{H}^T + \mathbf{R}]^{-1})$$

$$= \text{trace}(\mathbf{H}\mathbf{B}\mathbf{H}^T \mathbf{R}^{-1/2} [\mathbf{R}^{-1/2} \mathbf{H}\mathbf{B}\mathbf{H}^T \mathbf{R}^{-1/2} + \mathbf{I}]^{-1}) \mathbf{R}^{-1/2}$$

$$= \text{trace}(\mathbf{R}^{-1/2} \mathbf{H}\mathbf{B}\mathbf{H}^T \mathbf{R}^{-1/2} [\mathbf{R}^{-1/2} \mathbf{H}\mathbf{B}\mathbf{H}^T \mathbf{R}^{-1/2} + \mathbf{I}]^{-1}) \quad \text{trace } \mathbf{A}\mathbf{B} = \text{trace } \mathbf{B}\mathbf{A}$$

$$= \text{trace}(\mathbf{G}\mathbf{G}^T [\mathbf{G}\mathbf{G}^T + \mathbf{I}]^{-1})$$

where  $\mathbf{G} = \mathbf{R}^{-1/2} \mathbf{H}\mathbf{B}^{1/2}$  observability matrix

$$= \mathbf{E}\mathbf{\Gamma}^{1/2} (\mathbf{E}\mathbf{\Gamma}^{1/2})^T = \mathbf{E}_r \mathbf{\Gamma}_r \mathbf{E}_r^T$$

$$= \text{trace}(\mathbf{E}_r \mathbf{\Gamma}_r \mathbf{E}_r^T [\mathbf{E}_r \mathbf{\Gamma}_r \mathbf{E}_r^T + \mathbf{I}]^{-1})$$

# DFS (cont'd)

Using Sherman-Morrison-Woodbury formula

$$(\mathbf{A} + \mathbf{BDC})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}[\mathbf{D}^{-1} + \mathbf{CA}^{-1}\mathbf{B}]^{-1}\mathbf{CA}^{-1}$$

$$DFS = \text{trace}(\mathbf{E}_r \mathbf{\Gamma}_r \mathbf{E}_r^T [\mathbf{I} + \mathbf{E}_r \mathbf{\Gamma}_r \mathbf{E}_r^T]^{-1})$$

$$= \text{trace}(\mathbf{E}_r \mathbf{\Gamma}_r \mathbf{E}_r^T \{ \mathbf{I} - \mathbf{E}_r [\mathbf{\Gamma}_r^{-1} + \mathbf{E}_r^T \mathbf{E}_r]^{-1} \mathbf{E}_r^T \})$$

$$= \text{trace}(\mathbf{E}_r \{ \mathbf{\Gamma}_r \mathbf{E}_r^T - \cancel{\mathbf{\Gamma}_r \mathbf{E}_r^T} \mathbf{E}_r [\mathbf{\Gamma}_r^{-1} + \mathbf{I}]^{-1} \mathbf{E}_r^T \})$$

$$= \text{trace}(\mathbf{E}_r \{ \mathbf{\Gamma}_r - \mathbf{\Gamma}_r [\mathbf{\Gamma}_r^{-1} + \mathbf{I}]^{-1} \} \mathbf{E}_r^T)$$

$$= \text{trace}(\mathbf{\Gamma}_r - \mathbf{\Gamma}_r [\mathbf{\Gamma}_r^{-1} + \mathbf{I}]^{-1})$$

$$= \text{trace}(\mathbf{\Gamma}_r [\mathbf{\Gamma}_r^{-1} + \mathbf{I}]^{-1}) = \sum_{i=1}^r \frac{\gamma_i}{1 + \gamma_i}$$

**Thank you for your attention!**

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**Further information is available at**  
<https://kotsuki-lab.com/>

